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Scattering of surface water waves involving a vertical barrier with a gap

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Abstract. A mixed boundary-value problem associated with scattering of surface water waves by a vertical barrier with a gap of an arbitrary length is solved completely by the aid of the solution of a special logarithmic singular integral equation in the domain (a, b), which has bounded behaviour at both the end points $a(> 0)$ and b. The reflection coefficient is obtained analytically and its numerical values are presented graphically, for different values of the ratio of the width of the gap to the position of the gap. The present method of solution replaces the existing methods, which are either more elaborate or approximate in nature.

Key words: boundary-value problem, logarithmic kernels, reflection coefficient, singular integral equation, surface water waves.

1. Introduction

Mixed boundary-value problems arising in the theory of scattering of water waves by vertical barriers have been of interest to many research workers. A number of methods of solution have been explained for different barrier topographies by Ursell [1], Williams [2], Chakrabarti [3] and others (see Banerjea and Kar [4]). While Ursell's method of solution is based on the use of Havelock's expansion theorem and reduction of the various boundary-value problems to singular integral equations with Cauchy-type kernels, William's method, which is also called the 'reduction method', reduces these boundary-value problems to weakly singular integral equations having kernels with logarithmic singularities.

In the survey made on the various methods, available in the literature, Chakrabarti [3] has explained the major ideas in both the methods of Ursell [1] and Williams [2] and has also demonstrated a different method, by which the mixed boundary-value problems involving the two-dimensional Laplace equation occurring in water-wave scattering can be solved, which uses Abel-type integral equations and their solutions.

The solution of the problem of scattering of surface water waves by a vertical barrier with a gap was approximately solved by Tuck [5] in 1971. The complete analytical solution for this problem was first given by Porter [6] in 1972 by two different methods, in one of which the complex-variable method was used and, in the other, an integral-equation formulation with the help of the Green's function technique, both of which finally gave rise to a Riemann-Hilbert problem. The more general problem with a finite number of gaps in the barrier, was handled by Mei [7], by using complex-variable methods.

In the present paper, a simple and straightforward method is demonstrated to solve the boundary-value problem, arising in the scattering of surface water waves by a vertical barrier with a single gap, as was considered earlier by Porter [6] and Tuck [5]. The problem is reduced to a special logarithmic singular integral equation involving two unknown constants in the forcing function. This integral equation is solved completely by utilizing the boundedness property of the unknown function at the two endpoints of the interval in question.

The scattering problem is formulated in Section 2, and the reduction of the boundary-value problem into a special singular integral equation along with the behaviour of the unknown function in the integral equation, by using the edge conditions at the edges of the barrier under consideration, is demonstrated in Section 3. An analytical formula and numerical results for the reflection coefficient are determined by using appropriate constraints involving the solvability of the singular integral equation.

In Section 4, several aspects of the solution of the integral equation derived in Section 3, in particular limiting cases, are examined. The solution of the special logarithmic singular integral equation of Section 3, which is bounded at both the end points, is presented in the appendix.

The present method of solution of the particular boundary-value problem arising in the theory of scattering of surface water waves by barriers is exact and straightforward. Numerical values of quantities of practical importance are obtained by using standard procedures and these values are found to agree with the known ones.

2. Mathematical formulation

We consider the irrotational motion of an incompressible inviscid fluid under the action of gravity and use a rectangular Cartesian co-ordinate system in which the y-axis is taken vertically downward so that $y > 0$, $x \in I\!R$ is the region occupied by the fluid. The motion is two-dimensional and time-harmonic and is described by a velocity potential $\Phi(x, y, t)$ which is the real part of $\phi(x, y)e^{-i\omega t}$, $\omega(> 0)$ denoting angular frequency and t denoting the time. The time-dependent factor $e^{-i\omega t}$ is suppressed throughout the analysis. Then $\phi(x, y)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad x \in I\!R, y > 0, \tag{2.1}$$

which is a consequence of the equation of continuity: div $\vec{u} = 0$, $\vec{u}(=$ grad $\phi)$, denoting the velocity vector (see Stoker [8]).

Utilizing the continuity of pressure on the free surface of the fluid, we derive the linearised boundary condition

$$\frac{\partial\phi}{\partial y} + K\phi = 0, \text{ on } y = 0, x \in I\!R, \tag{2.2}$$

where $K = \omega^2/g$, with g being acceleration due to gravity. On the rigid vertical barrier occupied by $x = 0$, $y \in (0, a) \cup (b, \infty)$, ϕ satisfies the Neumann boundary condition

$$\frac{\partial\phi}{\partial x} = 0, \tag{2.3}$$

representing the condition of the vanishing of the normal velocity. Also, since the fluid flow is continuous across the gap (a, b), the velocity potential $\phi(x, y)$ satisfies

$$\phi(0^-, y) = \phi(0^+, y), \quad \text{for all } \ y \in (a, b), \tag{2.4}$$

in usual notations and

$$\frac{\partial\phi}{\partial x} \to 0, \;\; \frac{\partial\phi}{\partial y} \to 0 \text{ as } \; y \to \infty, \tag{2.5}$$

representing no motion as the depth of the fluid becomes large.

The behaviours of $\phi(x, y)$ at infinity in the horizontal direction, are given by

$$\phi(x, y) \to \begin{cases} e^{iKx - Ky} + R\, e^{-iKx - Ky}, & \text{as} \;\; x \to -\infty \\ T\, e^{iKx - Ky}, & \text{as} \;\; x \to \infty, \end{cases} \tag{2.6}$$

representing plane waves, where R and T are two unknown complex constants to be determined, of which R is the reflection coefficient and T is the transmission coefficient (see Stoker [8]).

The edge conditions, as required for the energy to be finite in the neighborhood of all edges associated with the flow (see [9, Section 2.4]), are given by

$$\frac{\partial\phi}{\partial x}(0, y) \sim O(|y - t|^{-\frac{1}{2}}) \text{ as } y \to t, \tag{2.7}$$

where $t = a^+$ and b^- are the edge points of the barrier under consideration.

3. The method of solution

3.1. Reduction to logarithmic singular integral equation

Following Williams [2], we reduce the boundary-value problem to a logarithmic singular integral equation as described below. We represent the unknown velocity potential $\phi(x, y)$, in the two regions $x < 0$ and $x > 0$, as given by,

$$\phi(x, y) = \begin{cases} T\, e^{iKx - Ky} + \int_0^\infty A(\xi)\, [\xi \cos \xi y - K \sin \xi y]\, e^{-\xi x}\, d\xi, \; x > 0, \\ e^{iKx - Ky} + R\, e^{-iKx - Ky} + \int_0^\infty B(\xi)\, [\xi \cos \xi y - K \sin \xi y]\, e^{\xi x}\, d\xi, \; x < 0, \end{cases} \tag{3.1}$$

where $A(\xi)$ and $B(\xi)$ are unknown functions to be determined, along with the unknown constants R and T, which are the reflection and the transmission coefficients of the incoming wave $e^{iKx - Ky}$. Note that the conservation of energy gives $|R|^2 + |T|^2 = 1$, and this will be verified by the numerical results obtained in the Subsection 3.3.

It is interesting to note that the above choice (3.1) of the function ϕ, automatically satisfies the partial differential equation (2.1) and the conditions (2.2), (2.5) and (2.6) for an appropriate choice of the functions $A(\xi)$ and $B(\xi)$, which will be decided as described below.

Since the horizontal velocity component is continuous across the positive y-axis, we get, by Havelock's expansion theorem that

$$T = 1 - R; \;\; A(\xi) = -B(\xi). \tag{3.2}$$

It can be easily seen that the conditions (2.3) and (2.4) along with the relations (3.2) give rise to a pair of integral equations as given by

$$\begin{aligned} \int_0^\infty \xi A(\xi)\, (\xi \cos \xi y - K \sin \xi y)\, d\xi \; &= \; iK(1 - R)\, e^{-Ky}, \;\; \text{on} \;\; y \in (0, a) \cup (b, \infty), \\ \int_0^\infty A(\xi)\, (\xi \cos \xi y - K \sin \xi y)\, d\xi \; &= \; R\, e^{-Ky}, \;\; \text{on} \;\; y \in (a, b), \end{aligned}$$

and these can be rewritten in an alternative form (see Chakrabarti [3]), as given by

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - K\right)\int_0^\infty \xi A(\xi)\ \sin\xi y\ \mathrm{d}\xi = \mathrm{i}K(1-R)\ \mathrm{e}^{-Ky},\ \text{ on } y \in (0,a)\cup(b,\infty), \tag{3.3}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - K\right)\int_0^\infty A(\xi)\ \sin\xi y\ \mathrm{d}\xi = R\ \mathrm{e}^{-Ky},\ \text{ on } y \in (a,b). \tag{3.4}$$

The above ordinary differential equations (3.3) and (3.4) can be easily solved to give the following dual integral equations:

$$\int_0^\infty \xi A(\xi)\ \sin\xi y\ \mathrm{d}\xi = \begin{cases} D_1\mathrm{e}^{Ky} + \mathrm{i}(1-R)\ \sinh Ky, & \text{for } y \in (0,a), \\ D_2\mathrm{e}^{Ky} - \frac{\mathrm{i}}{2}(1-R)\ \mathrm{e}^{-Ky}, & \text{for } y \in (b,\infty) \end{cases} \tag{3.5}$$

and

$$\int_0^\infty A(\xi)\ \sin\xi y\ \mathrm{d}\xi = E_1\mathrm{e}^{Ky} - \frac{R}{2K}\mathrm{e}^{-Ky},\ \text{ for } y \in (a,b), \tag{3.6}$$

where D_1, D_2 and E_1 are arbitrary constants.

In order to accommodate the origin, as well as the point at infinity along the y-axis, the arbitrary constants D_1 and D_2 in (3.5) are taken as zero. Then the dual integral equations (3.5–3.6) can be rewritten as

$$\int_0^\infty \xi A(\xi)\ \sin\xi y\ \mathrm{d}\xi = \begin{cases} \mathrm{i}(1-R)\ \sinh Ky, & \text{for } y \in (0,a), \\ -\frac{\mathrm{i}}{2}(1-R)\ \mathrm{e}^{-Ky}, & \text{for } y \in (b,\infty) \end{cases} \tag{3.7}$$

and

$$\int_0^\infty A(\xi)\ \sin\xi y\ \mathrm{d}\xi = E_1\mathrm{e}^{Ky} - \frac{R}{2K}\mathrm{e}^{-Ky},\ \text{ for } y \in (a,b). \tag{3.8}$$

Note that for the consistency of the relations (3.7) and (3.8), the constant E_1 must be taken as $R/2K$, when $a \to 0^+$ and zero, when $b \to \infty$. Now we define

$$\int_0^\infty \xi A(\xi)\ \sin\xi y\ \mathrm{d}\xi = g(y),\ \text{ for } y \in (a,b), \tag{3.9}$$

where $g(y)$ is an unknown function to be determined. Utilizing the relations (3.7) and (3.9), we obtain, by using the Fourier sine transform,

$$A(\xi) = \frac{2}{\pi\xi}\int_0^\infty P(y)\ \sin\xi y\ \mathrm{d}y, \tag{3.10}$$

where

$$P(y) = \begin{cases} \mathrm{i}(1-R)\ \sinh Ky, & \text{for } y \in (0,a), \\ g(y), & \text{for } y \in (a,b), \\ -\frac{\mathrm{i}}{2}(1-R)\ \mathrm{e}^{-Ky}, & \text{for } y \in (b,\infty). \end{cases}$$

By putting $A(\xi)$ into Equation (3.8) and after utilizing the result (see Gradshteyn and Ryzhik [10, Equation 3.741(1)])

$$\int_0^\infty \frac{\sin\xi y\ \sin\xi t}{\xi}\mathrm{d}\xi = -\frac{1}{2}\log\left|\frac{y-t}{y+t}\right|,\ \text{ for } y,t \in (0,\infty),$$

we obtain the following special logarithmic singular integral equation, to be solved for the unknown function $g(y)$, as given by

$$\frac{1}{\pi}\int_a^b g(u)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u = f(x), \text{ for } x\in(a,b), \tag{3.11}$$

where

$$f(x) = -\frac{\mathrm{i}(1-R)}{\pi}\int_0^a \sinh Kt\,\log\left|\frac{x+t}{x-t}\right|\mathrm{d}t + \frac{\mathrm{i}(1-R)}{2\pi}\int_b^\infty \mathrm{e}^{-Kt}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}t$$
$$+ E_1\mathrm{e}^{Kx} - \frac{R}{2K}\mathrm{e}^{-Kx},$$

with E_1 and R as two unknowns occurring in the forcing function.

In the next section, the integral equation (3.11) will be solved completely, and the unknown constants E_1 and R will also be determined.

3.2. Determining the reflection coefficient

In order to solve the integral equation (3.11) completely, we must know the behaviour of the unknown function $g(u)$ at the end points $u = a$ and $u = b$ which can be determined as follows. Letting

$$\frac{\partial\phi}{\partial x}(0,y) = h(y), \text{ for } y\in(a,b), \tag{3.12}$$

we have, from the relation (3.1), that

$$\left(\frac{\mathrm{d}}{\mathrm{d}y} - K\right)\int_0^\infty \xi\, A(\xi)\, \sin\xi y\,\mathrm{d}\xi = \mathrm{i}K(1-R)\,\mathrm{e}^{-Ky} - h(y), \text{ for } y\in(a,b). \tag{3.13}$$

The relation (3.13) is an ordinary differential equation in the domain (a,b) whose solution, along with the relation (3.9), gives rise to the following relation:

$$g(y) = C_1\mathrm{e}^{Ky} + \frac{\mathrm{i}(R-1)}{2}\mathrm{e}^{-Ky} - \mathrm{e}^{Ky}\int h(y)\mathrm{e}^{-Ky}\mathrm{d}y, \text{ for } y\in(a,b), \tag{3.14}$$

where C_1 is an arbitrary constant.
After differentiating, we can write the relation (3.14) as

$$\frac{\partial g}{\partial y} - Kg(y) = -h(y) - \mathrm{i}K(R-1)\,\mathrm{e}^{-Ky}, \text{ for } y\in(a,b). \tag{3.15}$$

Using the behaviour of the function $h(u)$ at the endpoints $u = a$ and $u = b$ from the relations (2.7), (3.12), we conclude that the unknown function $g(u)$, which is the solution of the above differential equation (3.15), is bounded at both the end points $u = a$ and $u = b$ and behaves as

$$g(y) \sim O(|y-t|^{\frac{1}{2}}) \text{ as } y\to t,$$

where $t = a^+$ and b^-.

It can be shown (see Appendix) that the solution of the integral equation (3.11), which is bounded at both the end points, is given by

$$g(u) = \frac{2}{\pi}\left\{(u^2-a^2)(b^2-u^2)\right\}^{\frac{1}{2}}\int_a^b \frac{tf'(t)}{\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}(u^2-t^2)}\mathrm{d}t, \; u\in(a,b), \tag{3.16}$$

provided that

$$(i)\ \int_a^b \frac{tf'(t)}{\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t = 0 \quad \text{and} \quad (ii)\ C + 2\int_a^b \left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}} tf'(t)\mathrm{d}t = 0 \tag{3.17}$$

where C is given by

$$C = 2\left(\frac{a\pi - J_1}{J_2}\right)\int_a^b \frac{f(x)}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{\frac{1}{2}}}\mathrm{d}x + 2\int_a^b \left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}} f(x)\mathrm{d}x,$$

with

$$J_1 = \int_a^b \left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}} \log\left|\frac{a+x}{a-x}\right|\mathrm{d}x, \quad J_2 = \int_a^b \frac{\log\left|\frac{a+x}{a-x}\right|}{\left\{(x^2-a^2)(b^2-x^2)\right\}^{\frac{1}{2}}}\mathrm{d}x.$$

Utilizing the function $f(x)$ of Equation (3.11) in the relations (3.17), we determine a linear system of equations, to be solved for the unknown constants R and E_1, which is given by

$$r_1R + a_1E_1 = b_1, \quad r_2R + a_2E_1 = b_2 \tag{3.18}$$

where

$$\begin{aligned}
r_1 = {} & -\mathrm{i}\int_0^a \frac{t\ \sinh Kt}{\left\{(a^2-t^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t - \frac{\mathrm{i}}{2}\int_b^\infty \frac{t\mathrm{e}^{-Kt}}{\left\{(t^2-a^2)(t^2-b^2)\right\}^{\frac{1}{2}}}\mathrm{d}t \\
& +\frac{1}{2}\int_a^b \frac{t\mathrm{e}^{-Kt}}{\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t, \\
a_1 = {} & K\int_a^b \frac{t\mathrm{e}^{Kt}}{\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t, \\
b_1 = {} & -\mathrm{i}\int_0^a \frac{t\ \sinh Kt}{\left\{(a^2-t^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t - \frac{\mathrm{i}}{2}\int_b^\infty \frac{t\mathrm{e}^{-Kt}}{\left\{(t^2-a^2)(t^2-b^2)\right\}^{\frac{1}{2}}}\mathrm{d}t, \\
r_2 = {} & \frac{\mathrm{i}}{\pi}\int_a^b \frac{(a\pi - J_1) + J_2(t^2-a^2)}{J_2\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\left[\int_0^a \sinh Kx\ \log\left|\frac{t+x}{t-x}\right|\mathrm{d}x\right]\mathrm{d}t \\
& -\frac{\mathrm{i}}{2\pi}\int_a^b \frac{(a\pi - J_1) + J_2(t^2-a^2)}{J_2\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\left[\int_b^\infty \mathrm{e}^{-Kx}\log\left|\frac{t+x}{t-x}\right|\mathrm{d}x\right]\mathrm{d}t \\
& -\frac{1}{2K}\int_a^b \frac{[(a\pi - J_1) + J_2(t^2-a^2)]\mathrm{e}^{-Kt}}{J_2\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t + \frac{1}{2}\int_a^b t\mathrm{e}^{-Kt}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}\mathrm{d}t \\
& +\mathrm{i}\int_0^a t\ \sinh Kt\left[\left(\frac{a^2-t^2}{b^2-t^2}\right)^{\frac{1}{2}} - 1\right]\mathrm{d}t - \frac{\mathrm{i}}{2}\int_b^\infty t\mathrm{e}^{-Kt}\left[\left(\frac{t^2-a^2}{t^2-b^2}\right)^{\frac{1}{2}} - 1\right]\mathrm{d}t. \\
a_2 = {} & \int_a^b \frac{[(a\pi - J_1) + J_2(t^2-a^2)]\mathrm{e}^{Kt}}{J_2\left\{(t^2-a^2)(b^2-t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t + K\int_a^b t\mathrm{e}^{Kt}\left(\frac{t^2-a^2}{b^2-t^2}\right)^{\frac{1}{2}}\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
b_2 = {} & \frac{\mathrm{i}}{\pi}\int_a^b \frac{(a\pi - J_1) + J_2(t^2 - a^2)}{J_2\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}}\left[\int_0^a \sinh Kx \ \log\left|\frac{t+x}{t-x}\right|\mathrm{d}x\right]\mathrm{d}t \\
& -\frac{\mathrm{i}}{2\pi}\int_a^b \frac{(a\pi - J_1) + J_2(t^2 - a^2)}{J_2\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}}\left[\int_b^\infty \mathrm{e}^{-Kx}\log\left|\frac{t+x}{t-x}\right|\mathrm{d}x\right]\mathrm{d}t \\
& +\mathrm{i}\int_0^a t\ \sinh Kt\left[\left(\frac{a^2 - t^2}{b^2 - t^2}\right)^{\frac{1}{2}} - 1\right]\mathrm{d}t - \frac{\mathrm{i}}{2}\int_b^\infty t\mathrm{e}^{-Kt}\left[\left(\frac{t^2 - a^2}{t^2 - b^2}\right)^{\frac{1}{2}} - 1\right]\mathrm{d}t.
\end{aligned}
$$

In order to determine the coefficients r_1, b_1, r_2 and b_2, we have utilized the following integrals which can be evaluated by an appropriate contour integration technique:

$$
(i)\quad \int_a^b \frac{t}{\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}(x^2 - t^2)}\mathrm{d}t = -\frac{\pi}{2\left\{(a^2 - x^2)(b^2 - x^2)\right\}^{\frac{1}{2}}},\quad \text{for } x < a,
$$

$$
(ii)\quad \int_a^b \frac{t}{\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}(x^2 - t^2)}\mathrm{d}t = \frac{\pi}{2\left\{(x^2 - a^2)(x^2 - b^2)\right\}^{\frac{1}{2}}},\quad \text{for } x > b,
$$

$$
(iii)\quad \int_a^b \left(\frac{t^2 - a^2}{b^2 - t^2}\right)^{\frac{1}{2}}\frac{t}{(x^2 - t^2)}\mathrm{d}t = \frac{2}{\pi}\left[\left(\frac{a^2 - x^2}{b^2 - x^2}\right)^{\frac{1}{2}} - 1\right],\quad \text{for } x < a \text{ and } x > b.
$$

We find that the constants R and E_1 satisfying the system (3.18) are given by the following relations:

$$
R = \frac{a_2 b_1 - a_1 b_2}{a_2 r_1 - a_1 r_2},\qquad E_1 = \frac{r_2 b_1 - r_1 b_2}{a_1 r_2 - a_2 r_1}. \tag{3.19}
$$

This completes the description involved in the determination of the analytical solution of the boundary value problem posed in Section 2. The final form of the solution $\phi(x, y)$ can be obtained by using the relations (3.2), (3.10), (3.16) and (3.19) in the relation (3.1).

3.3. Numerical results

The values of the constants $r_1, a_1, b_1, r_2, a_2, b_2$ as given in the relations (3.18) have been numerically computed by using *NIntegrate* in the *Mathematica*. Apparently, *NIntegrate* utilizes the Gauss quadrature rule for evaluating the integrals appearing in the coefficients $r_1, a_1, b_1, r_2, a_2, b_2$. The numerical computations for the reflection and transmission coefficients $|R|$ and $|T|$ have been carried out for different values of a non-dimensional parameter μ, as has been done in [6], which is given by the relations $\alpha = Kh$, $a = h(1 - \frac{\mu}{2})$ and $b = h(1 + \frac{\mu}{2})$, where h is the depth of the center of the gap below the free surface, giving μ as the ratio of the width of the gap to its mean depth (we note that $0 \leq \mu \leq 2$).

The graphical profiles of the reflection and transmission coefficients $|R|$ and $|T|$, for different values of the parameter μ, have been shown in Figure 1 and Figure 2, respectively, and these are observed to be comparable with Porter's [6] results shown in Figure 3. The interesting fact, as pointed out by Porter [6], can be seen from Figure 2, namely that for smaller values of the parameter μ, there is an appreciable amount of energy transmission at certain wave lengths. For example, when $\mu = 0{\cdot}1$, α is about 0·25, which is the situation of maximum transmission, when there is about 50% of wave-energy transmission.

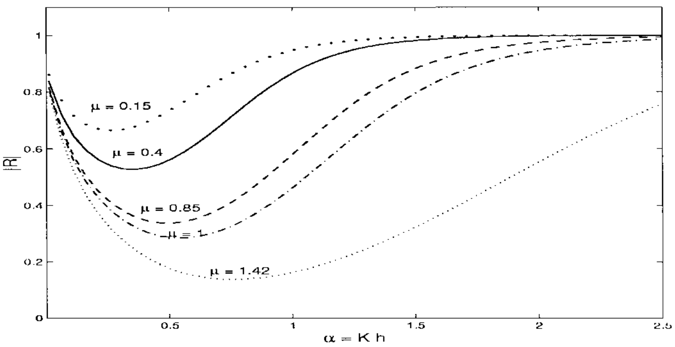

Figure 1. Reflection coefficient $|R|$ and $|T|$ for different values of the parameter μ.

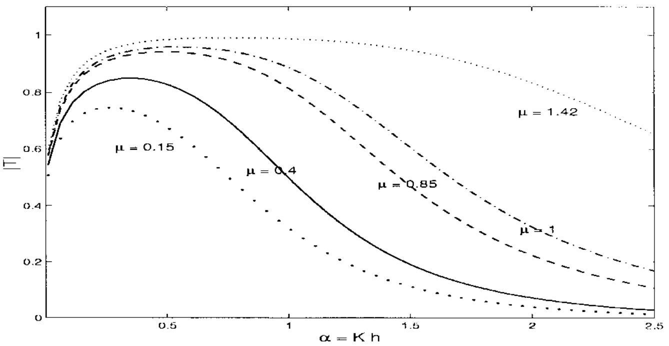

Figure 2. Transmission coefficient $|R|$ and $|T|$ for different values of the parameter μ.

It may be remarked from Figure 1 (Figure 2) that, for a given value of μ, the absolute value of the reflection (transmission) coefficient, $|R|$ ($|T|$), decreases (increases) for the low frequency of the incoming waves and, subsequently, the value of $|R|$ ($|T|$) increases (decreases) for waves of high frequency. Furthermore, the value of $|R|$ is close to unity for very low frequencies, as well as for very high ones. Since very low frequency waves are of very long wave length compared to the mean depth of the gap, these waves hardly 'feel' the gap, as was pointed out by Porter [6]. With an aim to study the sensitivity of $|R|$ ($|T|$) due to changes in μ, the values of $|R|$ ($|T|$) are computed for different values of μ (see Figures 1 and 2). It can be noted from the Figure 1 (Figure 2) that, as the value of μ decreases, which is the case of the gap being narrowed, the value of $|R|$ ($|T|$) increases (decreases), while the frequency of the incoming wave at which the value of $|R|$ ($|T|$) is minimized (maximized), decreases (increases). Finally, it has been observed numerically that the value of $|R|^2 + |T|^2$ is approximately equal to one, which verifies the conservation of energy.

4. Solution for other cases

In this section, the solution is considered in the limiting cases of the integral equation (3.11), having mathematical links with the problems associated with the gap giving rise to only a single portion of the barrier, instead of two portions in the general case.
Case (i): $a \to 0^+$ and $b(> 0)$ fixed.
In the limiting case when $a \to 0^+$, the integral equation (3.11) of Section-3, reduces to a special singular integral equation which is given by

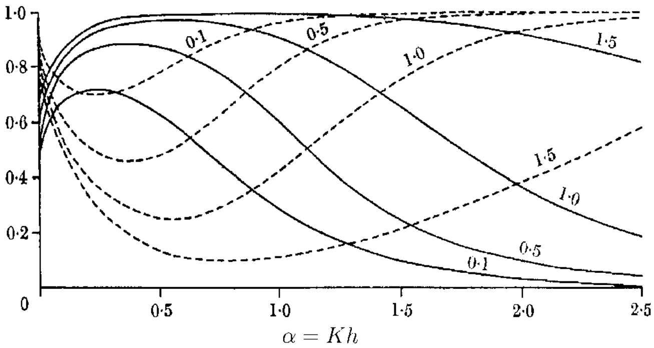

Figure 3. Reflection (- - -) and transmission (—) coefficients $|R|$ and $|T|$ for different values of the parameter μ.

$$\frac{1}{\pi}\int_0^b g(u)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u = f(x), \ \text{ for } x \in (0, b), \tag{4.1}$$

where

$$f(x) = \frac{\mathrm{i}(1-R)}{2\pi}\int_b^\infty \mathrm{e}^{-Kt}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}t + \frac{R}{K}\ \sinh Kx,$$

which is obtained, by choosing $E_1 = R/2K$ in the function $f(x)$ of the relation (3.11). In this case, the two conditions of solvability (3.17) are reduced to

$$(i)\ \int_0^b \frac{f'(t)}{(b^2-t^2)^{\frac{1}{2}}}\mathrm{d}t = 0, \quad \text{and } (ii)\ \int_0^b \frac{t^2 f'(t)}{(b^2-t^2)^{\frac{1}{2}}}\mathrm{d}t + \int_0^b \frac{tf(t)}{(b^2-t^2)^{\frac{1}{2}}}\mathrm{d}t = 0. \tag{4.2}$$

Since $f(0) = 0$, it can be easily observed, by replacing t^2 by $t^2 - b^2 + b^2$ in the first integral of the relation (4.2)(ii) and using integration by parts, that the above two conditions in the relation (4.2) are one and the same.

Utilizing the above condition (4.2)(i) and the following integrals (see Gradshteyn and Ryzhik [10, Equations 3.534(2) and 3.387(6)]),

$$(i)\int_0^b \frac{\cosh Kx}{(b^2-x^2)^{\frac{1}{2}}}\mathrm{d}x = \frac{\pi}{2}I_0(Kb), \ \text{ for } b, K > 0,$$

$$(ii)\int_0^b \frac{\mathrm{d}x}{(b^2-x^2)^{\frac{1}{2}}(y^2-x^2)} = \frac{\pi}{2y(y^2-b^2)^{\frac{1}{2}}}, \ \text{ for } y > b,$$

$$(iii)\int_b^\infty \frac{\mathrm{e}^{-Kx}}{(x^2-b^2)^{\frac{1}{2}}}\mathrm{d}x = K_0(Kb), \ \text{ for } b, K > 0,$$

where I_0, K_0 are the modified Bessel functions of the first and second kind, respectively, we find that the unknown constant R in the forcing function $f(x)$ of an integral equation (4.1) as given by

$$R = \frac{K_0(Kb)}{K_0(Kb) + i\pi I_0(Kb)}. \tag{4.3}$$

It is interesting to observe that the problem of scattering of surface water waves by a fully immersed barrier (see Ursell's problem, Ursell [1]) also gives rise to the integral equation (4.1) and the value of R, which represents the reflection coefficient in Ursell's problem agrees with the one as given by the relation (4.3).

We remark here that, even though the present problem in the limiting case $a \to 0^+$ does not correspond to the problem of Ursell [1], physically speaking, the integral equation (4.1) is common in both these problems, giving rise to the unknown value of R, showing the mathematical link between this limiting case and the fully immersed barrier problem of Ursell [1].

Case (ii): $a(>0)$ fixed and $b \to \infty$.
In this particular limiting case, the integral equation (3.11) reduces to

$$\frac{1}{\pi}\int_a^\infty g(u)\log\left|\frac{u+x}{u-x}\right|\mathrm{d}u = f(x), \quad \text{for } x \in (a,\infty), \tag{4.4}$$

where

$$f(x) = -\frac{\mathrm{i}(1-R)}{\pi}\int_0^a \sinh Kt\ \log\left|\frac{x+t}{x-t}\right|\mathrm{d}t - \frac{R}{2K}\mathrm{e}^{-Kx},$$

with R as an unknown constant occurring in the forcing function.

Transforming the above integral equation (4.4) into an integral equation of the form (4.1) and following *case (i),* we obtain the solvability criterion in this special case as

$$\int_a^\infty \frac{tf'(t)}{(t^2-a^2)^{\frac{1}{2}}}\mathrm{d}t = 0. \tag{4.5}$$

Using the above condition (4.5) and the following integrals (see Gradshteyn and Ryzhik [10, Equations (3.365(1,2) and 3.389(3)]),

$$(i)\int_0^a \frac{x\ \sinh Kx}{(a^2-x^2)^{\frac{1}{2}}}\mathrm{d}x = \frac{a\pi}{2}I_1(Ka), \quad \text{for } a, K > 0,$$

$$(ii)\int_a^\infty \frac{x}{(x^2-a^2)^{\frac{1}{2}}(y^2-x^2)}\mathrm{d}x = -\frac{\pi}{2(a^2-y^2)^{\frac{1}{2}}}, \quad \text{for } y < a,$$

$$(iii)\int_a^\infty \frac{x\ \mathrm{e}^{-Kx}}{(x^2-a^2)^{\frac{1}{2}}}\mathrm{d}x = a\ K_1(aK), \quad \text{for } a, K > 0,$$

where I_1, K_1, are again the modified Bessel functions, we obtain the unknown constant R in the forcing function $f(x)$ of the integral equation (4.4) as given by

$$R = \frac{\pi I_1(Ka)}{\pi I_1(Ka) + \mathrm{i}K_1(Ka)}.$$

It is observed that the above value of R agrees with the reflection coefficient corresponding to the problem of scattering by a finite surface piercing barrier as considered by Ursell [1]. Thus, as in the *case (i),* this limiting case also has, mathematically speaking a link with the surface-piercing barrier problem of Ursell [1].

5. Conclusions

A special new method of solution of an already solved problem of the scattering of surface water waves has been explained. The mixed boundary-value problem under consideration involves the two-dimensional Laplace equation, associated with the problem of the scattering of surface water waves by a vertical barrier, with a single gap in it. The problem is formulated in terms of a singular integral equation over the domain of the gap, with a logarithmic singularity in its kernel (see Equation (3.11)). The end behaviours of the unknown function of this integral equation are found to produce two mathematical constraints (see Equations (3.17)), and these constraints help in determining completely, all the unknowns associated with the problem. In particular, the most important physical quantity, known as the 'reflection coefficient', is determined analytically. The graphical profiles of the reflection and transmission coefficients have been plotted for different values of a non-dimensional parameter representing the ratio of the width of the gap to its mean depth and these are compared and found to be matching with the ones obtained by earlier workers. An extension of the present method of solution to the singular integral equation arising in the problems of the scattering of surface water waves involving vertical barriers with a finite number of gaps is also found to be possible and this will form the subject matter of a future publication.

Appendix

In this appendix, the solution of the integral equation (3.11) which is worked out in detail in [11], when f and g are differentiable functions, is briefly explained. Differentiating (see [12]) both sides of Equation (3.11), with respect to x, we obtain the following new integral equation for the function g :

$$\frac{1}{\pi}\int_a^b \frac{2ug(u)}{(u^2-x^2)}\mathrm{d}u = f'(x)(\equiv \frac{\mathrm{d}f}{\mathrm{d}x}),\ \ (a<x<b), \tag{A1}$$

whose general solution is well-known and is given by the following formula (see Gakhov [13, Chapter III])

$$g(u) = \frac{1}{\pi}\Big\{(u^2-a^2)(b^2-u^2)\Big\}^{-\frac{1}{2}}\Big[C + 2\int_a^b \frac{(t^2-a^2)(b^2-t^2)^{\frac{1}{2}} t f'(t)}{(u^2-t^2)}\mathrm{d}t\Big],\ \ (a<u<b), \tag{A2}$$

where C is an arbitrary constant. The constant C is connected with g and is given by the following relation:

$$C = 2\int_a^b u\ g(u)\mathrm{d}u. \tag{A3}$$

Utilizing the relations

$$\int_a^b \left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}} \log\left|\frac{u+x}{u-x}\right|\mathrm{d}x = \pi(u-a) + J_1,\ \ (a<u<b),$$

where

$$J_1 = \int_a^b \left(\frac{x^2-a^2}{b^2-x^2}\right)^{\frac{1}{2}} \log\left|\frac{a+x}{a-x}\right|\mathrm{d}x,$$

and

$$\int_a^b \frac{\log\left|\frac{u+x}{u-x}\right|}{\Big\{(x^2-a^2)(b^2-x^2)\Big\}^{\frac{1}{2}}}\mathrm{d}x = \int_a^b \frac{\log\left|\frac{a+x}{a-x}\right|}{\Big\{(x^2-a^2)(b^2-x^2)\Big\}^{\frac{1}{2}}}\mathrm{d}x = J_2,\ \ (a<u<b),$$

we determine the constant C, by using Equations (3.11) and (A3), as

$$C = 2\left(\frac{a\pi - J_1}{J_2}\right)\int_a^b \frac{f(x)}{\left\{(x^2 - a^2)(b^2 - x^2)\right\}^{\frac{1}{2}}}\mathrm{d}x + 2\int_a^b \left(\frac{x^2 - a^2}{b^2 - x^2}\right)^{\frac{1}{2}} f(x)\mathrm{d}x. \qquad \text{(A4)}$$

The solution (A2), with C given by the relation (A4) is the solution of the integral equation (3.11). Writing the solution (A2), in a rather formal manner, as given by

$$g(u) = \frac{1}{\pi}\left\{(u^2 - a^2)(b^2 - u^2)\right\}^{-\frac{1}{2}}\Bigg[C + (b^2 - u^2)\Bigg\{-2\int_a^b \frac{tf'(t)}{\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}}\mathrm{d}t$$
$$+2(u^2 - a^2)\int_a^b \frac{tf'(t)}{\left\{(t^2 - a^2)(b^2 - t^2)\right\}^{\frac{1}{2}}(u^2 - t^2)}\mathrm{d}t\Bigg\} + 2\int_a^b \left(\frac{t^2 - a^2}{b^2 - t^2}\right)^{\frac{1}{2}} tf'(t)\mathrm{d}t\Bigg],$$

for $a < u < b$,

we obtain solution (3.16) which is bounded at both the end points, of the integral equation (3.11) along with the two solvability conditions as given by the relations (3.17).

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